

New kinds of deformed Bessel functions

Mohammed Brahim Zahaf ^{a†} and Dominique Manchon ^{b‡}

^a *Laboratoire de Physique Quantique de la Matière et Modélisations Mathématiques (LPQ3M),
Université de Mascara, 29000-Mascara, Algérie*

^b *Laboratoire de Mathématiques, CNRS-UMR 6620
Université Blaise Pascal
BP 80026, 63171 Aubière Cedex,
France*

Abstract

Using a deformed calculus based on the Dunkl operator, two new deformations of Bessel functions are proposed. Some properties i.e. generating function, differential-difference equation, recursive relations, Poisson formula... are also given with detailed proofs. Three more deformations are also outlined in the last section.

Introduction

Dunkl operators on \mathbb{R}^n have been introduced in [3]: roughly speaking, they are partial derivative operators perturbed by reflexions. Although they are not differential operators in the usual sense, they mutually commute as classical partial derivative operators do. They have been intensively studied both from an algebraic [4, 5, 10] and analytic [11, 2] point of view.

In this article, we replace the derivation operator d/dx by the corresponding Dunkl operator:

$$D_\mu f(x) = \frac{d}{dx} f(x) + \mu \frac{f(x) - f(-x)}{x} \quad (0.1)$$

[†]E-mail address : m.b.zahaf@yahoo.fr

[‡]E-mail address : Dominique.Manchon@math.univ-bpclermont.fr

where μ is a real parameter, and consider the corresponding intertwining operator V_μ . It is the unique linear operator such that $V_\mu \circ (d/dx) = D_\mu \circ V_\mu$ and $V_\mu(1) = 1$ where 1 is the constant function equal to one. Applying this operator to the generating function of the classical Bessel functions J_n , we obtain a first family J_n^μ of deformed Bessel functions. They verify a differential-difference equation of order 3, as well as recursive formulas and a Poisson formula. They also verify a deformed version of the “addition theorem” (2.29), namely:

$$\tau_y J_n^\mu(x) = \sum_{k=-\infty}^{\infty} J_k^\mu(x) J_{n-k}^\mu(y), \quad (0.2)$$

where

$$\tau_y f := \sum_{n=0}^{\infty} \frac{y^n}{[n]_\mu!} D_\mu^n f, \quad (0.3)$$

with $[n]_\mu = n + \mu(1 - (-1)^n)$ and $[n]_\mu! = [n]_\mu [n-1]_\mu \cdots [1]_\mu$. Letting μ going to zero gives back the classical Bessel functions. We also give a second family \mathcal{J}_n^μ of deformed Bessel functions by directly modifying the Poisson formula (2.28). They also verify a differential-difference equation of order 3, but it seems there is no suitable addition theorem for them. Recursive relations are fulfilled, and the generating function is explicitly given. Moreover, a connection formula relating the \mathcal{J}_n^μ 's with their classical counterparts is easily given, whereas we haven't been able to give a similar formula for the first deformation.

We give three other families of deformed Bessel functions in the last section, together with the corresponding generating functions. They are obtained by directly deforming the coefficients of the Taylor series expansion of the J_n 's at the origin. Two among them verify recursive relations, but other properties of the classical Bessel functions do not seem to have their counterparts for these deformations.

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1 Dunkl operator

The Dunkl operator D_μ of index μ , $\mu \geq 0$, is defined on all smooth functions f on \mathbb{R} by

$$D_\mu f(x) = \frac{d}{dx} f(x) + \mu \frac{f(x) - f(-x)}{x}, \quad x \in \mathbb{R}. \quad (1.4)$$

For more general Dunkl operators see [5]. This operator has the following properties

$$D_\mu^2 f(x) = \frac{d^2}{dx^2} f(x) + \frac{2\mu}{x} \frac{d}{dx} f(x) - \frac{\mu}{x^2} (f(x) - f(-x)), \quad (1.5)$$

$$D_\mu(fg)(x) = f(x) D_\mu g(x) + g(-x) D_\mu f(x) + f'(x)(g(x) - g(-x)), \quad (1.6)$$

$$D_\mu x^n = [n]_\mu x^{n-1}, \quad n \in \mathbb{N}. \quad (1.7)$$

where $[n]_\mu = n + \mu(1 - (-1)^n)$. Obviously, $[2m]_\mu = 2m$, $[2m+1]_\mu = 2m+1+2\mu$ for any integer m , and when $\mu \rightarrow 0$, $[n]_\mu \rightarrow n$. Let us define the deformed exponential function by

$$E_\mu(x) = \sum_{n \geq 0} \frac{x^n}{[n]_\mu!}, \quad (1.8)$$

where $[n]_\mu! = [n]_\mu [n-1]_\mu \dots [1]_\mu$, $[0]_\mu! \equiv 1$. Then we have

$$D_\mu E_\mu(\lambda x) = \lambda E_\mu(x), \quad \lambda \in \mathbb{C}. \quad (1.9)$$

Recall the definition of the Pochhammer symbol:

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = a(a+1) \cdots (a+k-1). \quad (1.10)$$

Let us remark, using the following expressions [9]:

$$\begin{aligned} [2m]_\mu! &= \frac{2^{2m} m! \Gamma(m + \mu + \frac{1}{2})}{\Gamma(\mu + \frac{1}{2})} = (2m)! \frac{(\mu + \frac{1}{2})_m}{(\frac{1}{2})_m}, \\ [2m+1]_\mu! &= \frac{2^{2m+1} m! \Gamma(m + \mu + \frac{3}{2})}{\Gamma(\mu + \frac{1}{2})} = (2m+1)! \frac{(\mu + \frac{1}{2})_{m+1}}{(\frac{1}{2})_{m+1}}, \end{aligned} \quad (1.11)$$

that we can easily write $E_\mu(x)$ under the form

$$\begin{aligned} E_\mu(x) &= j_{\mu-\frac{1}{2}}(ix) + \frac{x}{2\mu+1} j_{\mu+\frac{1}{2}}(ix) \\ &= e^x {}_1F_1 \left(\begin{matrix} \mu \\ 2\mu+1 \end{matrix}, -2x \right), \end{aligned} \quad (1.12)$$

where j_α is the normalized spherical Bessel function defined for $\alpha \geq -\frac{1}{2}$, by

$$j_\alpha(x) = \sum_{k \geq 0} \frac{(-1)^k}{k! (\alpha+1)_k} \left(\frac{x}{2} \right)^{2k}.$$

For our purpose let us recall the following important theorem ([3], [4]): *There exists a unique linear isomorphism V_μ (called Dunkl intertwining operator) from the set of polynomials \mathcal{P}_n on \mathbb{R} of degree less or equal than n onto itself such that:*

$$V_\mu(1) = 1, \quad \text{and} \quad D_\mu V_\mu = V_\mu \frac{d}{dx}. \quad (1.13)$$

The operator V_μ has been extended by K. Trimèche to an isomorphism from $\mathcal{C}^\infty(\mathbb{R})$ onto itself satisfying the relations in (1.13) (see [11]). It possesses the following integral representation:

$$\forall x \in \mathbb{R}, \quad V_\mu(f(x)) = \frac{1}{\beta(\frac{1}{2}, \mu)} \int_{-1}^1 f(xt) (1-t)^{\mu-1} (1+t)^\mu dt, \quad f \in \mathcal{C}^\infty(\mathbb{R}). \quad (1.14)$$

We have

$$V_\mu(x^n) = \frac{\left(\frac{1}{2}\right)_{\lfloor \frac{n+1}{2} \rfloor}}{(\mu + \frac{1}{2})_{\lfloor \frac{n+1}{2} \rfloor}} x^n = \frac{n!}{[n]_\mu!} x^n, \quad (1.15)$$

where $\lfloor \alpha \rfloor$ stands for integer part of the real number α , and

$$E_\mu(x) = V_\mu(e^x). \quad (1.16)$$

The generalized translation operator τ_y , $y \in \mathbb{R}$ is defined by

$$\tau_y f := E_\mu(y D_\mu) f = \sum_{n=0}^{\infty} \frac{y^n}{[n]_\mu!} D_\mu^n f. \quad (1.17)$$

for all entire functions f on \mathbb{C} for which the series converges pointwise. It possesses the following properties:

$$\tau_y x^n = \sum_{k=0}^{+\infty} \frac{y^k}{[k]_\mu!} D_\mu^k x^n = \sum_{k=0}^{+\infty} \binom{n}{k}_\mu x^k y^{n-k}, \quad (1.18)$$

where $\binom{n}{k}_\mu = \frac{[n]_\mu!}{[k]_\mu! [n-k]_\mu!}$. We moreover have:

$$\tau_y E_\mu(\lambda x) = E_\mu(\lambda x) E_\mu(\lambda y), \quad \lambda \in \mathbb{C}. \quad (1.19)$$

2 Background on the classical Bessel functions

Let $n \in \mathbb{Z}$ be any integer. The classical Bessel function of order n is given by:

$$\begin{aligned} J_n(x) &= \sum_{k \geq 0} \frac{(-1)^k}{k!(k+n)!} \left(\frac{x}{2}\right)^{2k+n} \\ &= \frac{1}{n!} \left(\frac{x}{2}\right)^n {}_0F_1 \left(\begin{matrix} - \\ n+1 \end{matrix}, -\frac{x^2}{4} \right). \end{aligned} \quad (2.20)$$

In view of the relation:

$$J_{-n}(x) = (-1)^n J_n(x), \quad (2.21)$$

we shall consider only J_n for nonnegative n . The order n Bessel function is the solution of the following second-order linear differential equation:

$$\left(x^2 \frac{d^2}{dx^2} + x \frac{d}{dx} + (x^2 - n^2) \right) y(x) = 0. \quad (2.22)$$

with boundary conditions $J_n(0) = \delta_0^n$ and $\dot{J}_n(0) = \frac{1}{2}\delta_n^1$. The classical Bessel functions can be gathered into the generating function:

$$G(x, t) = \exp\left(\frac{x}{2}\left(t - \frac{1}{t}\right)\right) = \sum_{n=-\infty}^{+\infty} J_n(x)t^n. \quad (2.23)$$

The following recursive relations are satisfied:

$$2\dot{J}_n(x) = J_{n-1}(x) - J_{n+1}(x), \quad (2.24)$$

$$nJ_n(x) = xJ_{n-1}(x) - x\dot{J}_n(x), \quad (2.25)$$

$$nJ_n(x) = xJ_{n+1}(x) + x\dot{J}_n(x), \quad (2.26)$$

as well as

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x}J_n(x), \quad (2.27)$$

which can be obtained by adding (2.25) and (2.26). The *Poisson formula* is given by:

$$J_n(x) = \frac{\left(\frac{x}{2}\right)^n}{\Gamma(\frac{1}{2})\Gamma(n + \frac{1}{2})} \int_{-1}^1 (1 - s^2)^{n-\frac{1}{2}} \cos(xs) ds, \quad (2.28)$$

The following *addition theorem* holds:

$$J_n(x + y) = \sum_{k=-\infty}^{+\infty} J_k(x)J_{n-k}(y), \quad (2.29)$$

as one can easily see by using the generating function $G(x + y, t)$. For more details on classical Bessel functions, see for example [1, 7, 12].

3 First deformation of the Bessel function

We define the deformed Bessel function by

$$J_n^\mu(x) := V_\mu(J_n(x)), \quad (3.30)$$

where J_n is the classical Bessel function. In virtue of (1.15) we can write

$$J_n^\mu(x) = \sum_{k \geq 0} \frac{(-1)^k (2k + n)!}{k!(k + n)! [2k + n]_\mu!} \left(\frac{x}{2}\right)^{2k+n}. \quad (3.31)$$

Using the ratio test, we easily verify that this series converges in the whole complex plane, and hence represents an entire function of x . The deformed Bessel function of order n is a solution of the following differential-difference equation based on the deformed derivative (Dunkl operator):

$$\left((xD_\mu - [n]_\mu)(xD_\mu - [-n]_\mu)(xD_\mu + \beta_n + 2\mu - 1) + x^2(xD_\mu + \beta_n + 1)\right)y(x) = 0, \quad (3.32)$$

where

$$\beta_n = 2\lfloor \frac{n+1}{2} \rfloor - [n]_\mu = \begin{cases} 0 & \text{if } n \in 2\mathbb{N} \\ 1 - 2\mu & \text{if } n \in 2\mathbb{N} + 1 \end{cases} . \quad (3.33)$$

In fact,

$$\begin{aligned} & \left((xD_\mu - [n]_\mu)(xD_\mu - [-n]_\mu)(xD_\mu + \beta_n + 2\mu - 1) \right) J_n^\mu(x) = \\ &= \sum_{k \geq 0} \frac{(-1)^k (2k+n)!}{k!(k+n)! [2k+n]_\mu!} \left((2k + [n]_\mu - [n]_\mu)(2k + [n]_\mu - [-n]_\mu)(2k + [n]_\mu + \beta_n + 2\mu - 1) \right) \left(\frac{x}{2} \right)^{2k+n} \\ &= \sum_{k \geq 0} \frac{(-1)^k (2k+n)!}{k!(k+n)! [2k+n]_\mu!} \left((2k)(2k + 2n)(2k + 2\lfloor \frac{n+1}{2} \rfloor + 2\mu - 1) \right) \left(\frac{x}{2} \right)^{2k+n} \\ &= \sum_{k \geq 0} \frac{(-1)^k (2k+n)!}{(k-1)!(k+n-1)! [2k+n]_\mu!} \left(4(2k + 2\lfloor \frac{n+1}{2} \rfloor + 2\mu - 1) \right) \left(\frac{x}{2} \right)^{2k+n} \\ &= -x^2 \sum_{k \geq 0} \frac{(-1)^k (2k+n+2)!}{k!(k+n)! [2k+n+2]_\mu!} \left((2k + 2 + 2\lfloor \frac{n+1}{2} \rfloor + 2\mu - 1) \right) \left(\frac{x}{2} \right)^{2k+n} \\ &= -x^2 \sum_{k \geq 0} \frac{(-1)^k (2k+n)!}{k!(k+n)! [2k+n]_\mu!} \left(\frac{(2k+n+2)(2k+n+1)}{[2k+n+2]_\mu [2k+n+1]_\mu} (2k + 2\lfloor \frac{n+1}{2} \rfloor + 2\mu + 1) \right) \left(\frac{x}{2} \right)^{2k+n} \\ &= -x^2 \sum_{k \geq 0} \frac{(-1)^k (2k+n)!}{k!(k+n)! [2k+n]_\mu!} \left((2k + 2\lfloor \frac{n+1}{2} \rfloor + 1) \right) \left(\frac{x}{2} \right)^{2k+n} \\ &= - \left(x^2 (xD_\mu + \beta_n + 1) \right) J_n^\mu(x). \end{aligned}$$

It is easy to see, when $\mu = 0$, the third order deformed differential equation (3.32) reduces to the second order Bessel differential equation (2.22). We have also:

$$\begin{aligned} J_{-n}^\mu(x) &= \sum_{k \geq n} \frac{(-1)^k (2k-n)!}{k!(k-n)! [2k-n]_\mu!} \left(\frac{x}{2} \right)^{2k-n} \\ &= \sum_{s \geq 0} \frac{(-1)^{s+n} (2s+n)!}{(s+n)! s! [2s+n]_\mu!} \left(\frac{x}{2} \right)^{2s+n}, \end{aligned}$$

and hence:

$$J_{-n}^\mu(x) = (-1)^n J_n^\mu(x), \quad n = 1, 2, \dots \quad (3.34)$$

The generating function of the deformed Bessel function is given by:

$$G^\mu(x, t) = E_\mu \left(\frac{x}{2} \left(t - \frac{1}{t} \right) \right) = \sum_{n=-\infty}^{+\infty} J_n^\mu(x) t^n. \quad (3.35)$$

This is obtained by applying the intertwining operator V_μ to the generating function $G(x, t)$ for the classical Bessel function, with respect to the variable x , and using the relation (1.16). For $t = 1$, we obtain the following relation:

$$\sum_{n=-\infty}^{+\infty} J_n^\mu(x) = 1, \quad (3.36)$$

which can be also written as:

$$J_0^\mu(x) + 2 \sum_{n=1}^{+\infty} J_{2n}^\mu(x) = 1. \quad (3.37)$$

If we take $t = e^{i\theta}$ in (3.35) we obtain:

$$E_\mu(ix \sin \theta) = \sum_{n=-\infty}^{+\infty} J_n^\mu(x) e^{in\theta}. \quad (3.38)$$

This implies that:

$$J_n^\mu(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} E_\mu(ix \sin \theta) e^{-in\theta} d\theta. \quad (3.39)$$

The deformed Bessel function possesses the following recurrence relations:

$$2D_\mu J_n^\mu(x) = J_{n-1}^\mu(x) - J_{n+1}^\mu(x), \quad (3.40)$$

$$nD_\mu J_n^\mu(x) = \frac{d}{dx} (xJ_{n-1}^\mu(x) - xD_\mu J_n^\mu(x)), \quad (3.41)$$

$$nD_\mu J_n^\mu(x) = \frac{d}{dx} (xJ_{n+1}^\mu(x) + xD_\mu J_n^\mu(x)). \quad (3.42)$$

Summing up (3.41) and (3.42) we obtain:

$$2nD_\mu J_n^\mu(x) = \frac{d}{dx} (xJ_{n-1}^\mu(x) + xJ_{n+1}^\mu(x)). \quad (3.43)$$

The three last equations (3.41), (3.42) and (3.43) are not as simple as their classical counterparts (2.25), (2.26) and (2.27), due to the fact that the deformed derivative D_μ differs from the ordinary derivative d/dx . The first relation is obtained by applying the Dunkl operator to the left and the right hand of (3.35). For the second one, we have:

$$\begin{aligned} \frac{d}{dx} (xJ_{n-1}^\mu(x) - xD_\mu J_n^\mu(x)) &= \sum_{k \geq 0} \frac{(-1)^k (2k+n)!}{k!(k+n-1)! [2k+n-1]_\mu!} \left(\frac{x}{2}\right)^{2k+n-1} \\ &\quad - \sum_{k \geq 0} \frac{(-1)^k (2k+n)!(2k+n)}{k!(k+n)! [2k+n-1]_\mu!} \frac{1}{2} \left(\frac{x}{2}\right)^{2k+n-1} \\ &= n \sum_{k \geq 0} \frac{(-1)^k (2k+n)!}{k!(k+n)! [2k+n-1]_\mu!} \frac{1}{2} \left(\frac{x}{2}\right)^{2k+n-1} \\ &= nD_\mu J_n^\mu(x). \end{aligned}$$

The third one follows similarly.

A formula involving deformed Bessel functions with different *superscripts* is

$$(xD_\mu + \beta_n + 2\mu - 1)J_n^\mu(x) = (2\mu - 1)J_n^{\mu-1}(x), \quad (3.44)$$

where β_n is given in (3.33). To prove this equality we use the fact

$$(a)_k = \frac{a+k-1}{a-1}(a-1)_k, \quad (3.45)$$

which leads to:

$$\begin{aligned} \frac{(2k+n)!}{[2k+n]_\mu!} &= \frac{(\frac{1}{2})_{k+\lfloor \frac{n+1}{2} \rfloor}}{(\mu + \frac{1}{2})_{k+\lfloor \frac{n+1}{2} \rfloor}} \\ &= \frac{(\frac{1}{2})_{k+\lfloor \frac{n+1}{2} \rfloor}}{(\mu - \frac{1}{2})_{k+\lfloor \frac{n+1}{2} \rfloor}} \frac{\mu - \frac{1}{2}}{k + \lfloor \frac{n+1}{2} \rfloor + \mu - \frac{1}{2}} \\ &= \frac{(2k+n)!}{[2k+n]_{\mu-1}!} \frac{\mu - \frac{1}{2}}{k + \lfloor \frac{n+1}{2} \rfloor + \mu - \frac{1}{2}}, \end{aligned}$$

and therefore:

$$\begin{aligned} (xD_\mu + \beta_n + 2\mu - 1)J_n^\mu(x) &= \sum_{k \geq 0} \frac{(-1)^k (2k+n)!}{k!(k+n)! [2k+n]_\mu!} (2k + [n]_\mu + \beta_n + 2\mu - 1) \left(\frac{x}{2}\right)^{2k+n} \\ &= \sum_{k \geq 0} \frac{(-1)^k (2k+n)!}{k!(k+n)! [2k+n]_\mu!} (2k + 2\lfloor \frac{n+1}{2} \rfloor + 2\mu - 1) \left(\frac{x}{2}\right)^{2k+n} \\ &= (2\mu - 1) \sum_{k \geq 0} \frac{(-1)^k (2k+n)!}{k!(k+n)! [2k+n]_{\mu-1}!} \left(\frac{x}{2}\right)^{2k+n} \\ &= (2\mu - 1)J_n^{\mu-1}(x). \end{aligned}$$

Using the fact:

$$\frac{(2k+n)!}{[2k+n]_\mu!} = \frac{(\frac{1}{2})_{k+\lfloor \frac{n+1}{2} \rfloor}}{(\mu + \frac{1}{2})_{k+\lfloor \frac{n+1}{2} \rfloor}} = \frac{(\frac{1}{2})_{\lfloor \frac{n+1}{2} \rfloor}}{(\mu + \frac{1}{2})_{\lfloor \frac{n+1}{2} \rfloor}} \frac{(\lfloor \frac{n+1}{2} \rfloor + \frac{1}{2})_k}{(\lfloor \frac{n+1}{2} \rfloor + \mu + \frac{1}{2})_k}, \quad (3.46)$$

the deformed Bessel function can be expressed in terms of the generalized hypergeometric function ${}_1F_2$ as follows:

$$\begin{aligned} J_n^\mu(x) &= \frac{n!}{[n]_\mu!} \sum_{k \geq 0} \frac{(-1)^k}{k!(k+n)!} \frac{(\lfloor \frac{n+1}{2} \rfloor + \frac{1}{2})_k}{(\lfloor \frac{n+1}{2} \rfloor + \mu + \frac{1}{2})_k} \left(\frac{x}{2}\right)^{2k+n} \\ &= \frac{1}{[n]_\mu!} \left(\frac{x}{2}\right)^n \sum_{k \geq 0} \frac{(\lfloor \frac{n+1}{2} \rfloor + \frac{1}{2})_k}{k!(n+1)_k (\lfloor \frac{n+1}{2} \rfloor + \mu + \frac{1}{2})_k} \left(-\frac{x^2}{4}\right)^k \\ &= \frac{1}{[n]_\mu!} \left(\frac{x}{2}\right)^n {}_1F_2 \left(\begin{matrix} \lfloor \frac{n+1}{2} \rfloor + \frac{1}{2} \\ n+1, \lfloor \frac{n+1}{2} \rfloor + \mu + \frac{1}{2} \end{matrix} ; -\frac{x^2}{4} \right). \end{aligned} \quad (3.47)$$

By applying the intertwining operator V_μ to the left and right hand of the classical Bessel function poisson formula (2.28), we obtain the μ -version of poisson formula

$$J_n^\mu(x) = \frac{n!}{[n]_\mu!} \frac{\left(\frac{x}{2}\right)^n}{\Gamma(\frac{1}{2})\Gamma(n+\frac{1}{2})} \int_{-1}^1 (1-s^2)^{n-\frac{1}{2}} {}_1F_2\left(\begin{matrix} \lfloor \frac{n+1}{2} \rfloor + \frac{1}{2} \\ \frac{1}{2} \end{matrix} \middle| \begin{matrix} \lfloor \frac{n+1}{2} \rfloor + \mu + \frac{1}{2} \end{matrix} , -\frac{(sx)^2}{4} \right) ds. \quad (3.48)$$

The deformed Bessel function $J_n^\mu(x)$ verifies the following analogue of the addition theorem (2.29):

$$\tau_y J_n^\mu(x) = \sum_{k=-\infty}^{\infty} J_k^\mu(x) J_{n-k}^\mu(y). \quad (3.49)$$

This follows immediately from the fact that:

$$\tau_y E_\mu\left(\frac{x}{2}\left(t - \frac{1}{t}\right)\right) = E_\mu\left(\frac{x}{2}\left(t - \frac{1}{t}\right)\right) E_\mu\left(\frac{y}{2}\left(t - \frac{1}{t}\right)\right),$$

or equivalently:

$$\sum_{n=-\infty}^{+\infty} \tau_y J_n^\mu(x) t^n = \sum_{n=-\infty}^{+\infty} J_n^\mu(x) t^n \sum_{n=-\infty}^{+\infty} J_n^\mu(y) t^n.$$

The result in (3.49) is obtained by equating the coefficients of t^n .

The deformed Bessel functions of order n have the following asymptotic form, as $|x| \rightarrow \infty$ and $|\arg(x)| \leq \frac{\pi}{2} - \varepsilon$ for some $\varepsilon > 0$:

$$\begin{aligned} J_n^\mu(x) &\approx \frac{\Gamma(\mu + \frac{1}{2})}{\pi} \left(\frac{x}{2}\right)^{-\mu-\frac{1}{2}} \cos\left(x - \frac{\pi}{2}\left(\frac{1}{2} + n + \mu\right)\right) \\ &+ \frac{\Gamma(\lfloor \frac{n+1}{2} \rfloor + \frac{1}{2})\Gamma(\mu + \frac{1}{2})}{\Gamma(n + \frac{1}{2} - \lfloor \frac{n+1}{2} \rfloor)\Gamma(\mu)\Gamma(\frac{1}{2})} \left(\frac{x}{2}\right)^{n-2\lfloor \frac{n+1}{2} \rfloor-1}, \end{aligned} \quad (3.50)$$

which is obtained by using the asymptotic formula of the generalized hypergeometric function ${}_1F_2$:

$$\begin{aligned} {}_1F_2\left(\begin{matrix} \alpha \\ n+1 \end{matrix} \middle| \begin{matrix} \beta \end{matrix} , -x^2\right) &\approx \frac{\Gamma(\beta)n!}{\Gamma(\frac{1}{2})\Gamma(\alpha)} x^{-\frac{1}{2}-n+\alpha-\beta} \cos\left(2x - \frac{\pi}{2}\left(\frac{1}{2} + n + \beta - \alpha\right)\right) \\ &+ \frac{\Gamma(\beta)n!}{\Gamma(\beta-\alpha)\Gamma(n+1-\alpha)} x^{-2\alpha}, \end{aligned} \quad (3.51)$$

as $|x| \rightarrow \infty$ and $|\arg(x)| \leq \frac{\pi}{2} - \varepsilon$ for some $\varepsilon > 0$, which is a special case of a general formula given by Luke [8, p. 203, Eq. (4)].

4 A second deformation of the Bessel function

We define a second deformation of Bessel function by using an other deformation of the poisson formula:

$$\mathcal{J}_n^\mu(x) := \frac{\left(\frac{x}{2}\right)^n}{\Gamma(\frac{1}{2})\Gamma(n+\frac{1}{2})} \int_{-1}^1 E_\mu(isx)(1-s^2)^{n-\frac{1}{2}} ds. \quad (4.52)$$

Using the series representation of $E_\mu(isx)$ we can easily write the function $\mathcal{J}_n^\mu(x)$ as:

$$\mathcal{J}_n^\mu(x) = \sum_{k \geq 0} \frac{(-1)^k (2k)!}{k!(k+n)! [2k]_\mu!} \left(\frac{x}{2}\right)^{2k+n}, \quad (4.53)$$

which can also written in terms of the generalized hypergeometric function ${}_1F_2$ as:

$$\mathcal{J}_n^\mu(x) = \frac{1}{n!} \left(\frac{x}{2}\right)^n {}_1F_2 \left(\begin{matrix} \frac{1}{2} \\ n+1 \quad \mu + \frac{1}{2} \end{matrix}, -\frac{x^2}{4} \right). \quad (4.54)$$

We remark that $\mathcal{J}_0^\mu(x) = J_0^\mu(x)$. The deformed Bessel function $\mathcal{J}_n^\mu(x)$ is a solution of the deformed differential equation:

$$\left((xD_\mu - [n]_\mu)(xD_\mu - [-n]_\mu)(xD_\mu - [n]_\mu + 2\mu - 1) + x^2(xD_\mu - [n]_\mu + 1) \right) y(x) = 0. \quad (4.55)$$

In fact,

$$\begin{aligned} & \left((xD_\mu - [n]_\mu)(xD_\mu - [-n]_\mu)(xD_\mu - [n]_\mu + 2\mu - 1) \right) \mathcal{J}_n^\mu(x) = \\ &= \sum_{k \geq 0} \frac{(-1)^k (2k)!}{k!(k+n)! [2k]_\mu!} \left((2k + [n]_\mu - [n]_\mu)(2k + [n]_\mu - [-n]_\mu)(2k + [n]_\mu - [n]_\mu + 2\mu - 1) \right) \left(\frac{x}{2}\right)^{2k+n} \\ &= \sum_{k \geq 0} \frac{(-1)^k (2k)!}{k!(k+n)! [2k]_\mu!} \left((2k)(2k + 2n)(2k + 2\mu - 1) \right) \left(\frac{x}{2}\right)^{2k+n} \\ &= \sum_{k \geq 0} \frac{(-1)^k (2k)!}{(k-1)!(k+n-1)! [2k]_\mu!} \left(4(2k + 2\mu - 1) \right) \left(\frac{x}{2}\right)^{2k+n} \\ &= -x^2 \sum_{k \geq 0} \frac{(-1)^k (2k+2)!}{k!(k+n)! [2k+2]_\mu!} \left((2k + 2 + 2\mu - 1) \right) \left(\frac{x}{2}\right)^{2k+n} \\ &= -x^2 \sum_{k \geq 0} \frac{(-1)^k (2k)!}{k!(k+n)! [2k]_\mu!} \left(\frac{(2k+2)(2k+1)}{[2k+2]_\mu [2k+1]_\mu} (2k + 2\mu + 1) \right) \left(\frac{x}{2}\right)^{2k+n} \\ &= -x^2 \sum_{k \geq 0} \frac{(-1)^k (2k)!}{k!(k+n)! [2k]_\mu!} (2k + 1) \left(\frac{x}{2}\right)^{2k+n} \\ &= -x^2 (xD_\mu - [n]_\mu + 1) \mathcal{J}_n^\mu(x). \end{aligned}$$

The generating function of the deformed Bessel functions $\mathcal{J}_n^\mu(x)$ for $n \geq 0$ is given by:

$$\mathcal{G}^\mu(x, t) = e^{\frac{xt}{2}} {}_1F_1 \left(\begin{matrix} \frac{1}{2} \\ \mu + \frac{1}{2} \end{matrix}, -\frac{x}{2t} \right) = \mathcal{J}_0^\mu(x) + \sum_{n=1}^{+\infty} \mathcal{J}_n^\mu(x) (t^n + (-1)^n t^{-n}). \quad (4.56)$$

In fact, let us take

$$e^{\frac{xt}{2}} {}_1F_1 \left(\begin{matrix} \frac{1}{2} \\ \mu + \frac{1}{2} \end{matrix}, -\frac{x}{2t} \right) = \sum_{n=-\infty}^{+\infty} c_n(x) t^n. \quad (4.57)$$

To calculate the coefficients $c_n(x)$, we multiply the power series

$$e^{\frac{xt}{2}} = \sum_{n \geq 0} \frac{\left(\frac{xt}{2}\right)^n}{n!}, \quad (4.58)$$

$${}_1F_1\left(\frac{1}{2}, \mu + \frac{1}{2}, -\frac{x}{2t}\right) = \sum_{n \geq 0} \frac{\left(\frac{1}{2}\right)_n}{n!(\mu + \frac{1}{2})_n} \left(-\frac{x}{2t}\right)^n, \quad (4.59)$$

and then combine terms containing identical powers of t . As a result, we obtain

$$\begin{aligned} c_n &= \mathcal{J}_n^\mu(x), \quad n = 0, 1, 2, \dots, \\ c_n &= (-1)^n \mathcal{J}_{-n}^\mu(x), \quad n = -1, -2, \dots, \end{aligned} \quad (4.60)$$

which implies (4.56). The deformed Bessel function $\mathcal{J}_n^\mu(x)$ possesses the following recursive relations

$$xD_\mu \mathcal{J}_n^\mu(x) = x\mathcal{J}_{n-1}^\mu(x) + [-n]_\mu \mathcal{J}_n^\mu(x), \quad (4.61)$$

$$\left(x\frac{d}{dx} - n\right) \left(\mathcal{J}_{n+1}^\mu(x) + \mathcal{J}_{n-1}^\mu(x) - \frac{2n-2\mu}{x} \mathcal{J}_n^\mu(x)\right) = -\frac{2\mu}{x} \mathcal{J}_n^\mu(x). \quad (4.62)$$

Indeed, for the first one we have:

$$\begin{aligned} (xD_\mu - [-n]_\mu) \mathcal{J}_n^\mu(x) &= \sum_{k \geq 0} \frac{(-1)^k (2k)! (2k+2n)}{k!(k+n)! [2k]_\mu!} \left(\frac{x}{2}\right)^{2k+n} \\ &= x \sum_{k \geq 0} \frac{(-1)^k (2k)!}{k!(k+n-1)! [2k]_\mu!} \left(\frac{x}{2}\right)^{2k+n-1} \\ &= x \mathcal{J}_{n-1}^\mu(x). \end{aligned}$$

For the second one, we have:

$$\begin{aligned} &\mathcal{J}_{n+1}^\mu(x) + \mathcal{J}_{n-1}^\mu(x) - \frac{2n-2\mu}{x} \mathcal{J}_n^\mu(x) = \\ &= \sum_{k \geq 0} \frac{(-1)^k (2k)!}{k!(k+n)! [2k]_\mu!} \left\{ -\frac{k[2k-1]_\mu [2k]_\mu}{(2k-1)(2k)} + n + k - n + \mu \right\} \left(\frac{x}{2}\right)^{2k+n-1} \\ &= \sum_{k \geq 0} \frac{(-1)^k (2k)!}{k!(k+n)! [2k]_\mu!} \left\{ -\frac{\mu}{2k-1} \right\} \left(\frac{x}{2}\right)^{2k+n-1}, \end{aligned}$$

and after applying the operator $x\frac{d}{dx} - n$, we easily retrieve the relation (4.62).

Using the fact that:

$${}_1F_1\left(\begin{matrix} a \\ b \end{matrix}; x\right) = e^x {}_1F_1\left(\begin{matrix} b-a \\ b \end{matrix}; -x\right) \quad (4.63)$$

we can write:

$$\mathcal{G}^\mu(x, t) = G(x, t) {}_1F_1\left(\begin{matrix} \mu \\ \mu + \frac{1}{2} \end{matrix}; \frac{x}{2t}\right) \quad (4.64)$$

which gives a connection formula between the deformed Bessel function and the classical one:

$$\mathcal{J}_n^\mu(x) = \sum_{k \geq 0} \frac{(\mu)_k}{k!(\mu + \frac{1}{2})_k} \left(\frac{x}{2}\right)^k J_{n+k}(x). \quad (4.65)$$

As above we can prove the following formulae involving deformed Bessel functions $\mathcal{J}_n^\mu(x)$ with different *superscripts* :

$$(xD_\mu - [n]_\mu + 2\mu - 1)\mathcal{J}_n^\mu(x) = (2\mu - 1)\mathcal{J}_n^{\mu-1}(x), \quad (4.66)$$

as well as:

$$\mathcal{J}_{n-1}^\mu(x) - \mathcal{J}_{n+1}^\mu(x) + \frac{2\mu}{2\mu+1}\mathcal{J}_{n+1}^{\mu+1}(x) = 2\frac{d}{dx}\mathcal{J}_n^\mu(x), \quad (4.67)$$

$$\mathcal{J}_{n-1}^\mu(x) + \mathcal{J}_{n+1}^\mu(x) - \frac{2\mu}{2\mu+1}\mathcal{J}_{n+1}^{\mu+1}(x) = \frac{2n}{x}\mathcal{J}_n^\mu(x). \quad (4.68)$$

The function $\mathcal{J}_n^\mu(x)$ has the following asymptotic form, as $|x| \rightarrow \infty$ and $|\arg(x)| \leq \frac{\pi}{2} - \varepsilon$ for some $\varepsilon > 0$:

$$\begin{aligned} \mathcal{J}_n^\mu(x) &\approx \frac{\Gamma(\mu + \frac{1}{2})}{\pi} \left(\frac{x}{2}\right)^{-\mu-\frac{1}{2}} \cos\left(x - \frac{\pi}{2}\left(\frac{1}{2} + n + \mu\right)\right) \\ &+ \frac{\Gamma(\mu + \frac{1}{2})}{\Gamma(n + \frac{1}{2})\Gamma(\frac{1}{2})} \left(\frac{x}{2}\right)^{-1}. \end{aligned} \quad (4.69)$$

5 Other possibilities of deformation

5.1 Three other versions of the deformed Bessel functions

We give three more deformations of the classical Bessel functions, with partial analogues for the set of formulae (2.24) to (2.27). A complete analogue of all formulae together does not seem to be available.

First deformation:

$$J_n^{(1,\mu)}(x) = \sum_{k \geq 0} \frac{(-1)^k}{k![k+n]_\mu!} \left(\frac{x}{2}\right)^{2k+n}. \quad (5.70)$$

$$\text{Generating function : } e^{\frac{xt}{2}} E_\mu \left(-\frac{x}{2t} \right) = \sum_{n=-\infty}^{+\infty} J_n^{(1,\mu)}(x) t^n. \quad (5.71)$$

$$\text{Recursive relation: } x D_\mu J_n^{(1,\mu)}(x) = -x J_{n+1}^{(1,\mu)}(x) + [n]_\mu J_n^{(1,\mu)}(x) \quad (5.72)$$

Second deformation:

$$J_n^{(2,\mu)}(x) = \sum_{k \geq 0} \frac{(-1)^k}{[k]_\mu! (k+n)!} \left(\frac{x}{2} \right)^{2k+n}. \quad (5.73)$$

$$\text{Generating function : } E_\mu \left(\frac{xt}{2} \right) e^{-\frac{x}{2t}} = \sum_{n=-\infty}^{+\infty} J_n^{(2,\mu)}(x) t^n. \quad (5.74)$$

$$\text{Recursive relation: } x D_\mu J_n^{(2,\mu)}(x) = x J_{n-1}^{(2,\mu)}(x) - n J_n^{(2,\mu)}(x) \quad (5.75)$$

Third deformation:

$$J_n^{(3,\mu)}(x) = \sum_{k \geq 0} \frac{(-1)^k}{[k]_\mu! [k+n]_\mu!} \left(\frac{x}{2} \right)^{2k+n}. \quad (5.76)$$

$$\text{Generating function: } E_\mu \left(\frac{xt}{2} \right) E_\mu \left(-\frac{x}{2t} \right) = \sum_{n=-\infty}^{+\infty} J_n^{(3,\mu)}(x) t^n. \quad (5.77)$$

6 Appendix: hypergeometric functions

The generalized hypergeometric function ${}_pF_q$ is defined by the series

$${}_pF_q \left(\begin{matrix} a_1 & a_2 & \dots & a_p \\ b_1 & b_2 & \dots & b_q \end{matrix} ; x \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \dots (a_p)_n}{(b_1)_n (b_2)_n \dots (b_q)_n} \frac{x^n}{n!} \quad (6.78)$$

It can be shown that the series converges for all x if $p \leq q$, converges for $|x| < 1$ if $p = q + 1$, and diverges for all $x \neq 0$ if $p > q + 1$. It is a solution of the differential equation:

$$\left(x \frac{d}{dx} + a_1 \right) \dots \left(x \frac{d}{dx} + a_p \right) y - \frac{d}{dx} \left(x \frac{d}{dx} + b_1 - 1 \right) \dots \left(x \frac{d}{dx} + b_q - 1 \right) y = 0. \quad (6.79)$$

The following differential recursive equations hold:

$$\begin{aligned}
\left(x \frac{d}{dx} + a_1\right) {}_pF_q \left(\begin{matrix} a_1 & a_2 & \dots & a_p \\ b_1 & b_2 & \dots & b_q \end{matrix} ; x \right) &= a_1 {}_pF_q \left(\begin{matrix} a_1 + 1 & a_2 & \dots & a_p \\ b_1 & b_2 & \dots & b_q \end{matrix} ; x \right), \\
\left(x \frac{d}{dx} + b_1 - 1\right) {}_pF_q \left(\begin{matrix} a_1 & a_2 & \dots & a_p \\ b_1 & b_2 & \dots & b_q \end{matrix} ; x \right) &= (b_1 - 1) {}_pF_q \left(\begin{matrix} a_1 & a_2 & \dots & a_p \\ b_1 - 1 & b_2 & \dots & b_q \end{matrix} ; x \right), \\
\frac{d}{dx} {}_pF_q \left(\begin{matrix} a_1 & a_2 & \dots & a_p \\ b_1 & b_2 & \dots & b_q \end{matrix} ; x \right) &= \frac{a_1 \dots a_p}{b_1 \dots b_q} {}_pF_q \left(\begin{matrix} a_1 + 1 & a_2 + 1 & \dots & a_p + 1 \\ b_1 + 1 & b_2 + 1 & \dots & b_q + 1 \end{matrix} ; x \right).
\end{aligned}$$

Special cases of hypergeometric functions are for example:

$$e^x = {}_0F_0 \left(\begin{matrix} - \\ - \end{matrix} ; x \right), \quad (6.80)$$

$$(1 - x)^{-a} = {}_1F_0 \left(\begin{matrix} a \\ - \end{matrix} ; x \right), \quad (6.81)$$

The Bessel function of order ν can be expressed in two different ways as a hypergeometric function:

$$J_\nu(x) = \frac{(x/2)^\nu}{\Gamma(\nu + 1)} {}_0F_1 \left(\begin{matrix} - \\ \nu + 1 \end{matrix} ; -\frac{x^2}{4} \right) \quad (6.82)$$

$$= \frac{e^{-ix}(x/2)^\nu}{\Gamma(\nu + 1)} {}_1F_1 \left(\begin{matrix} \nu + 1/2 \\ 2\nu + 1 \end{matrix} ; 2ix \right). \quad (6.83)$$

For a detailed account, see for example [6, 7], or any textbook on special functions.

References

- [1] M. Abramowitz, I. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, National Bureau of Standards Applied Mathematics Series **55** Washington, D.C. (1964).
- [2] S. Ben Saïd, B. Ørsted, *The wave equation for Dunkl operators*, Indag. Math. (N.S.) **16** (2005), no. 3-4, 351-391.
- [3] C. F. Dunkl, *Integral kernels with reflection group invariance*, Can. J. Math **43** (1991), 1213-1227.
- [4] C. F. Dunkl, M. F. E. De Jeu, E. M. Opdam, *Singular polynomials for finite reflection groups*, Trans. Amer. Math. Soc. **346** (1994), 237-256.
- [5] C. F. Dunkl, E. M. Opdam, *Dunkl operators for complex reflection groups*, Proc. London Math. Soc. **86** No3 (2003), 70-108.
- [6] A. Erdélyi, W. Magnus, F. Oberhettinger, F. G. Tricomi, *Higher transcendental functions I, II*, McGraw-Hill Book Company, Inc., New York-Toronto-London, 1953.

- [7] N. N. Lebedev, *Special functions and their applications*, Prentice-Hall (1965) (Translated from Russian).
- [8] Y. L. Luke, *The Special Functions and Their Approximations*, Vol. I, Academic Press, New York and London (1969).
- [9] M. Rosenblum, *Generalized Hermite polynomials and the Bose-like oscillator calculus*, in *Nonselfadjoint Operators and Related Topics* (Beer Sheva, 1992), Oper. Theory Adv. Appl., Vol. **73**, Birkhäuser, Basel (1994), 369-396.
- [10] Ch. Torossian, *Deux résultats autour du théorème de restriction de Chevalley*, J. Lie Theory **17** (2007), no. 3, 583-590.
- [11] K. Trimèche, *The Dunkl intertwining operator on spaces of functions and distributions and integral representation of its dual*. Integral Transforms Spec. Funct. **12** No 4 (2001), 349-374.
- [12] G. N. Watson, *A Treatise on the Theory of Bessel Function*, Cambridge University Press, Cambridge (1952).